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Many avatars of the Wilson fermion: A perturbative analysis

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ABSTRACT: We explore different branches of the fermion doublers with Wilson fermion in perturbation theory, in the context of additive mass renormalization and chiral anomaly, and show that by appropriately averaging over suitably chosen branches one can reduce cut-off artifacts. Comparing the central branch with all other branches, we find that the central branch, among all the avatars of the Wilson fermion, is the most suitable candidate for exploring near conformal lattice field theories.

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1 Introduction

The most reliable techniques to investigate various non-perturbative aspects of quantum field theories are provided by lattice methods. Putting fermion on a lattice, however, has turned out to be highly non-trivial because of the notorious doubling problem. Naive discretization of the Dirac action leads to 16 solutions (called doublers) in the four dimensional theory. Among the various solutions suggested to cure this problem, Wilson fermions [1] are conceptually the simplest and straightforward to implement. In the Wilson approach a dimension five operator is added to the action thereby sending the masses of the extra fifteen fermions to the order of the cutoff. Thus the extra fermions decouple in the continuum limit. It is well-known that the Wilson formulation of lattice gauge theory preserves discrete symmetries of the continuum formulation which simplifies the construction of lattice operators that correspond to the observables in the continuum theory. The Wilson term in the action, which is introduced to remove the doublers, breaks chiral symmetry in accordance with the well-known Nielsen-Ninomiya no-go theorem [2]. Wilson term reproduces the correct axial anomaly [3–5] even though it leads to additive renormalization for the fermion mass.

The sixteen doublers are classified into five branches. Almost all of the studies so far, both analytical and numerical, have focused on the so-called first (physical) branch. However, very recently, occurrence of an enhanced symmetry has been discovered in the central branch [6–8] when the on-site terms (mass term and that from the Wilson term) are absent in the action. The enhanced symmetry prohibits additive renormalization through radiative corrections. Since in this case, the central branch yields six massless fermions, as suggested by ref. [7], an alternative way to simulate twelve flavour non-abelian gauge theories emerges. Such theories are of interest in the context of beyond standard model physics (for recent reviews, see [9–12]).

In this work, by introducing a generalized Wilson term containing a branch selector index, we investigate the additive fermion mass shift and chiral anomaly to $\mathcal{O}(g^2)$ in lattice perturbation theory for all the branches of the fermion doublers.

2 Preliminaries

We denote the generalized Wilson fermion action by

$$S_F[\psi, \bar{\psi}, U](i_B) = a^4 \sum_{x,y} \bar{\psi}_x M_{xy}(i_B) \psi_y = a^4 \sum_{x,y} \bar{\psi}_x [\gamma_\mu D_\mu + W(i_B) + m]_{xy} \psi_y \quad \text{with} \quad (2.1)$$

$$[D_\mu]_{xy} = \frac{1}{2a} \left[U_{x,\mu} \delta_{x+\mu,y} - U_{x-\mu,\mu}^\dagger \delta_{x-\mu,y} \right] \quad \text{and} \quad (2.2)$$

$$W_{xy}(i_B) = \frac{r}{2a} \sum_\mu \left[2(1 - \frac{1}{2}i_B) \delta_{x,y} - U_{x,\mu} \delta_{x+\mu,y} - U_{x-\mu,\mu}^\dagger \delta_{x-\mu,y} \right]. \quad (2.3)$$

The factor i_B is the branch selector index which takes the values 0, 1, 2, 3, and 4 for first, second, third (central), fourth and fifth branch of the doubler respectively. We consider the transformations [7]

$$\psi_x \rightarrow \psi'_x = e^{i\theta(-1)^{x_1+x_2+x_3+x_4}} \psi_x, \quad \bar{\psi}_x \rightarrow \bar{\psi}'_x = e^{i\theta(-1)^{x_1+x_2+x_3+x_4}} \bar{\psi}_x. \quad (2.4)$$

The action is invariant under these transformations but for the local terms. For $m = 0$ and $i_B = 2$ (massless limit of the central branch), the action thus possesses this additional symmetry which prevents additive renormalization of the fermion mass through radiative corrections.

3 Additive Renormalization in Fermion Self Energy

In this section we calculate the additive shift to $\mathcal{O}(g^2)$ in the fermion mass (for $am = 0$) using lattice perturbation theory [13].

The additive shift in fermion mass due to the tadpole diagram (figure 1) is

$$\delta m = -\frac{r}{a} \frac{1}{2} g^2 C_F \sum_\mu \cos(ap_\mu) Z_0 \quad (3.1)$$

with $Z_0 = \int \frac{d^4 k}{(2\pi)^4} \left(4 \sum_\lambda \sin^2 \left(\frac{ak_\lambda}{2} \right) \right)^{-1}$ and $C_F = \frac{N^2-1}{2N}$ for SU(N).

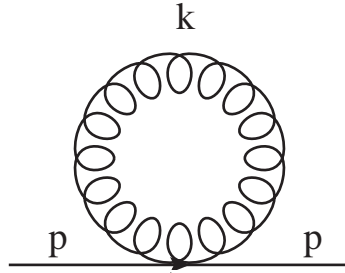


Figure 1. Tadpole diagram

Results for different branches are as follows.

First branch: $ap_\mu = (0, 0, 0, 0) \rightarrow \delta m = -2 \frac{r}{a} g^2 C_F Z_0$.

Second branch: $ap_\mu = (\pi, 0, 0, 0)$ or any of the other three permutations $\rightarrow \delta m = -1 \frac{r}{a} g^2 C_F Z_0$.

Third (central) branch: $ap_\mu = (\pi, \pi, 0, 0)$ or any of the other five permutations $\rightarrow \delta m = 0$.

Fourth branch: $ap_\mu = (\pi, \pi, \pi, 0)$ or any of the other three permutations $\rightarrow \delta m = +1 \frac{r}{a} g^2 C_F Z_0$.

Fifth branch: $ap_\mu = (\pi, \pi, \pi, \pi) \rightarrow \delta m = +2 \frac{r}{a} g^2 C_F Z_0$.

Next consider the additive mass shift in fermion mass due to sunset diagram. The gauge boson propagator in Feynman gauge is given by

$$G_{\mu\nu}^{ab} = \delta_{\mu\nu} \delta^{ab} \left\{ \frac{4}{a^2} \sum_{\lambda} \sin^2 \frac{a(p-k)_{\lambda}}{2} \right\}^{-1} = \delta_{\mu\nu} \delta^{ab} \left\{ (1/a^2) \mathcal{W}_{p,k} \right\}^{-1}, \quad (3.2)$$

whereas the fermion propagator has the form

$$S^{lm}(k, i_B) = \delta^{lm} \left\{ \sum_{\mu} i \gamma_{\mu} \frac{\sin(k_{\mu} a)}{a} + \frac{r}{a} \mathcal{M}_k(i_B) \right\}^{-1} \quad (3.3)$$

with

$$\mathcal{M}_k(i_B) = \sum_{\mu} \left[\left(1 - \frac{1}{2} i_B \right) - \cos(k_{\mu} a) \right] \quad (3.4)$$

and the fermion-gauge boson vertex is

$$(V^a)_{\rho}^{mn}(k, p) = -g(T^a)^{mn} \left\{ i \gamma_{\rho} \cos \frac{a(k+p)_{\rho}}{2} + r \sin \frac{a(k+p)_{\rho}}{2} \right\}. \quad (3.5)$$

Then the fermion self energy from sunset diagram can be evaluated as

$$\Sigma = \int \frac{d^4 k}{(2\pi)^4} \sum_{\rho} G_{\rho\rho}^{ab}(p-k) (V^b)_{\rho}^{lm}(k, p) S^{mn}(k) (V^a)_{\rho}^{nl}(p, k). \quad (3.6)$$

The additive mass shift arising from the fermion self energy (sunset) can be written as

$$\delta m = \frac{r}{a} g^2 C_F \int \frac{d^4 k}{(2\pi)^4} \frac{N_r}{D_r} \quad (3.7)$$

where $D_r = \mathcal{W}_{p,k}(\Gamma^2 + r^2 \mathcal{M}_k^2(i_B))$ with $\Gamma^2 = \sum_{\lambda} \sin^2(ak_{\lambda})$. We introduce $\Gamma_{\lambda} = \sin(ak_{\lambda})$, $S_{\rho} = \sin(\frac{ak_{\rho}}{2})$ and $C_{\rho} = \cos(\frac{ak_{\rho}}{2})$. The expressions for N_r and D_r for different branches are given below.

First branch: $ap_{\mu} = (0, 0, 0, 0)$.

$$N_r = \sum_{\rho=1}^4 \left[\mathcal{M}_k(i_B=0)(r^2 S_{\rho}^2 - C_{\rho}^2) + \Gamma_{\rho}^2 \right], \quad (3.8)$$

$$D_r = \mathcal{W}_{p,k} \left[\Gamma^2 + r^2 \mathcal{M}_k^2(i_B=0) \right]. \quad (3.9)$$

Second branch: $ap_{\mu} = (\pi, 0, 0, 0)$ or three other permutations. Explicitly for $ap_{\mu} = (\pi, 0, 0, 0)$

$$N_r = - \left[\mathcal{M}_k(i_B=1)(S_{\rho}^2 - r^2 C_{\rho}^2) + \Gamma_{\rho}^2 \right]_{\rho=1} + \sum_{\rho=2}^4 \left[\mathcal{M}_k(i_B=1)(r^2 S_{\rho}^2 - C_{\rho}^2) + \Gamma_{\rho}^2 \right], \quad (3.10)$$

$$D_r = \mathcal{W}_{p,k} \left[\Gamma^2 + r^2 \mathcal{M}_k^2(i_B=1) \right]. \quad (3.11)$$

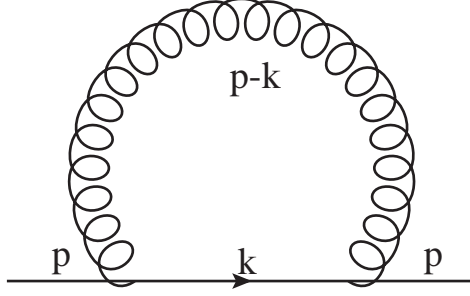


Figure 2. Sunset diagram

Third (central) branch: $ap_\mu = (\pi, \pi, 0, 0)$ or five other permutations.

Explicitly for $ap_\mu = (\pi, \pi, 0, 0)$

$$N_r = - \sum_{\rho=1}^2 \left[\mathcal{M}_k(i_B=2)(S_\rho^2 - r^2 C_\rho^2) + \Gamma_\rho^2 \right] + \sum_{\rho=3}^4 \left[\mathcal{M}_k(i_B=2)(r^2 S_\rho^2 - C_\rho^2) + \Gamma_\rho^2 \right], \quad (3.12)$$

$$D_r = \mathcal{W}_{p,k} \left[\Gamma^2 + r^2 \mathcal{M}_k^2(i_B=2) \right]. \quad (3.13)$$

Fourth branch: $ap_\mu = (\pi, \pi, \pi, 0)$ or three other permutations. Explicitly for $ap_\mu = (\pi, \pi, \pi, 0)$

$$N_r = - \sum_{\rho=1}^3 \left[\mathcal{M}_k(i_B=3)(S_\rho^2 - r^2 C_\rho^2) + \Gamma_\rho^2 \right] + \left[\mathcal{M}_k(i_B=3)(r^2 S_\rho^2 - C_\rho^2) + \Gamma_\rho^2 \right]_{\rho=4}, \quad (3.14)$$

$$D_r = \mathcal{W}_{p,k} \left[\Gamma^2 + r^2 \mathcal{M}_k^2(i_B=3) \right]. \quad (3.15)$$

Fifth branch: $ap_\mu = (\pi, \pi, \pi, \pi)$.

$$N_r = - \sum_{\rho=1}^4 \left[\mathcal{M}_k(i_B=4)(S_\rho^2 - r^2 C_\rho^2) + \Gamma_\rho^2 \right], \quad (3.16)$$

$$D_r = \mathcal{W}_{p,k} \left[\Gamma^2 + r^2 \mathcal{M}_k^2(i_B=4) \right]. \quad (3.17)$$

In table 1 we present the numerical values of the additive mass shift separately from sunset and tadpole contributions for the fermion at different branches. In figure 3 we plot the magnitude of the total additive mass shift (tadpole + sunset) versus the branch number. Note that the shift is maximum for the first and the fifth branches and is minimum (zero) for the third (central) branch. The absence of additive renormalization in fermion self energy for the central branch to $\mathcal{O}(g^2)$ is explicitly shown also in ref. [8].

4 Chiral Anomaly

Now we consider the flavor singlet axial Ward Identity

$$\langle \Delta_\mu^b J_{5\mu}(x) \rangle = 2m \langle \bar{\psi}_x \gamma_5 \psi_x \rangle + \langle \chi_x \rangle \quad (4.1)$$

Branch	$\delta m / (g^2 C_F)$	
	Sunset	Tadpole
First	-0.0158	-0.3099
Second	+0.0148	-0.1549
Third	0.0000	0.0000
Fourth	-0.0148	+0.1549
Fifth	+0.0158	+0.3099

Table 1. Numerical values of the additive mass shift for fermion at different branches for $r = 1$ and $L = 200$.

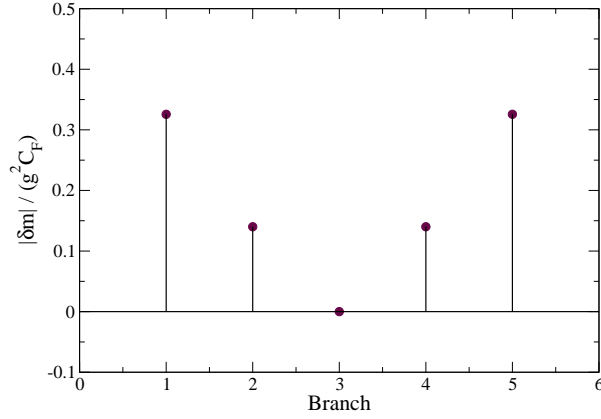


Figure 3. The magnitude of the total additive mass shift (tadpole + sunset) plotted versus the branch number.

where $\langle \mathcal{O} \rangle$ denotes the functional average of \mathcal{O} . Explanation of other terms are as follows:

$$\Delta_\mu^b f(x) = \frac{1}{a} [f(x) - f(x - \mu)] , \text{ the backward derivative,} \quad (4.2)$$

$$J_{5\mu}(x) = \frac{1}{2} \left[\bar{\psi}_x \gamma_\mu \gamma_5 U_{x,\mu} \psi_{x+\mu} + \bar{\psi}_{x+\mu} \gamma_\mu \gamma_5 U_{x\mu}^\dagger \psi_x \right] , \text{ the axial vector current} \quad (4.3)$$

$$\begin{aligned} \text{and } \langle \chi_x \rangle &= 2 g^2 \varepsilon_{\mu\nu\rho\lambda} \text{Trace } F_{\mu\nu}(x) F_{\rho\lambda}(x) \frac{1}{(2\pi)^4} \sum_p \cos(p_\mu a) \cos(p_\nu a) \cos(p_\rho a) \\ &\quad \times r \mathcal{M}_p(i_B) \left[\cos(p_\lambda a) [am + r \mathcal{M}_p(i_B)] - 4r \sin^2(p_\lambda a) \right] (\mathcal{G}_p(i_B))^3, \\ &= -\frac{g^2}{16\pi^2} \varepsilon_{\mu\nu\rho\lambda} \text{Trace } F_{\mu\nu}(x) F_{\rho\lambda}(x) I(am, r, L). \end{aligned} \quad (4.4)$$

Here

$$\mathcal{G}_p(i_B) = \left(\sum_\mu \sin^2(ap_\mu) + [am + r \mathcal{M}_p(i_B)]^2 \right)^{-1}. \quad (4.5)$$

Explicitly, $\sum_p = (\frac{2\pi}{L})^4 \sum_{n_1, n_2, n_3, n_4}$. In all our plots it is the anomaly integral denoted by the function $I(am, r, L)$ which we have plotted.

Following Karsten and Smit [3], the limits on the momentum sum are changed from $(-\pi, +\pi)$ to $(-\pi/2, 3\pi/2)$ and further the momentum sum hypercube is divided into 16 smaller hypercubes corresponding to $(-\pi/2, +\pi/2)$ and $(+\pi/2, +3\pi/2)$ for each $ap_\mu, \mu = 1, 2, 3, 4$. Thus the total anomaly contribution is decomposed into the contributions from five different types of species and the anomaly integral takes the form $I = I_0 - 4I_1 + 6I_2 - 4I_3 + I_4$. In I_0 all the four momentum integrations range from $(-\pi/2, +\pi/2)$ and in I_4 they range from $(+\pi/2, +3\pi/2)$. In I_1 one of the momentum integrations range from $(+\pi/2, +3\pi/2)$, the rest from $(-\pi/2, +\pi/2)$ and vice-versa for I_3 . In the third (central) branch I_2 two momentum integrations range from $(+\pi/2, +3\pi/2)$ and the rest from $(-\pi/2, +\pi/2)$.

First, to perform the integration analytically, we set the bare mass $am = 0$, use the identity [3]

$$\begin{aligned} \left[\mathcal{M}_p(i_B) \right]^2 \cos(ap_\beta) - 4r \mathcal{M}_p(i_B) \sin^2(ap_\beta) &= \left[\left[\mathcal{M}_p(i_B) \right]^2 + \sum_\sigma \sin^2(ap_\sigma) \right]^3 \times \\ &\quad \frac{\partial}{\partial(ap_\beta)} \left[\sin(ap_\beta) \left\{ \left[\mathcal{M}_p(i_B) \right]^2 + \sum_\sigma \sin^2(ap_\sigma) \right\} \right]^{-2} \end{aligned} \quad (4.6)$$

and do a partial integration. In the infinite volume continuum limit, the results for the integrals are as follows.

First branch: $I_0 \rightarrow 1, I_1, I_2, I_3, I_4 \rightarrow 0$.

Second branch: $I_1 \rightarrow 1, I_0, I_2, I_3, I_4 \rightarrow 0$.

Third (central) branch: $I_2 \rightarrow 1, I_0, I_1, I_3, I_4 \rightarrow 0$.

Fourth branch: $I_3 \rightarrow 1, I_0, I_1, I_2, I_4 \rightarrow 0$.

Fifth branch: $I_4 \rightarrow 1, I_0, I_1, I_2, I_3 \rightarrow 0$.

Since numerical simulations are performed at finite volume and finite lattice spacing, it is of interest to study the effect of symmetry violation on the anomaly integral as a function of the lattice fermion mass at finite volume [14, 15]. In order to avoid the zero mode problem we have used antiperiodic boundary condition in all four directions. In figure 4 we plot the function $I(am, r, L)$ for $r = 1.0$ and $L = 100$ as a function of am for the first and fifth branches (left) and for the second and fourth branches (right). In figure 5, we plot the function $I(am, r, L)$ for $r = 1.0$ and $L = 100$ as a function of am for the central branch. From figure 4 (left), we observe that the cut-off effects are almost equal and opposite for first and fifth branches. Similar observation can be made regarding second and fourth branches from figure 4 (right). Comparing figures 4 and 5, we conclude that cut-off effects are minimal for the central branch. We have picked $L = 100$ for our plots as we have found that finite volume effects are negligible at this volume for the range of am shown in the figures.

5 Discussion and Conclusions

It is well known that the naive discretization of the fermionic action gives rise to sixteen degenerate species including the desired physical one. These sixteen species are grouped into five branches with degeneracy (chirality) given by 1(1), 4(-1), 6(1), 4(-1) and 1(1), rendering the theory free of

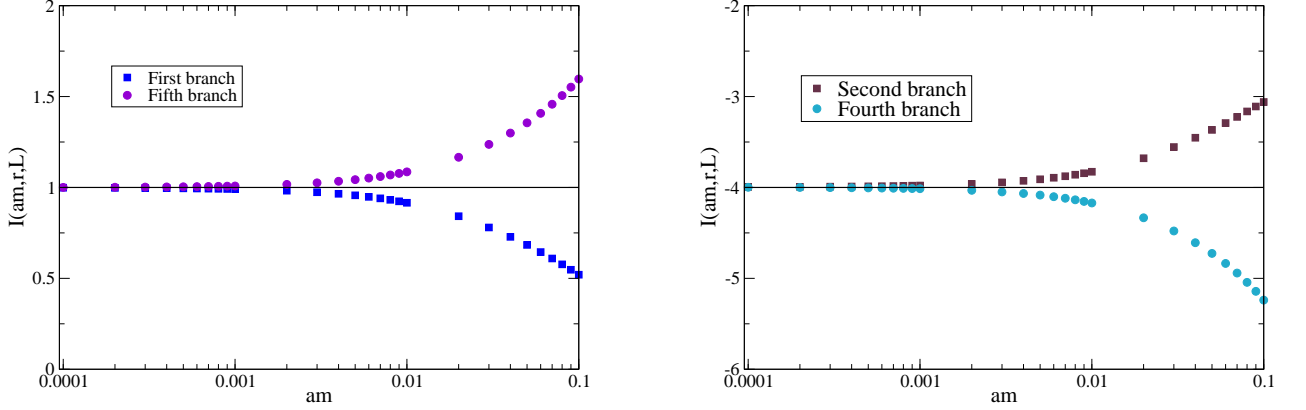


Figure 4. The function $I(am, r, L)$ for $r = 1.0$ and $L = 100$ as a function of am for the first and fifth branches (left) and for the second and fourth branches (right).

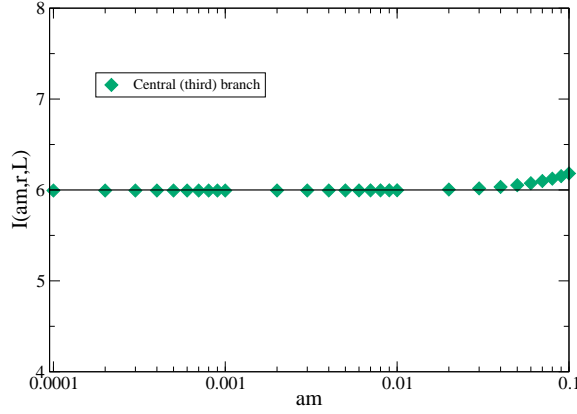


Figure 5. The function $I(am, r, L)$ for the central branch for $r = 1.0$ and $L = 100$ as a function of am .

chiral anomaly. With the conventional Wilson term in the continuum limit, apart from the first branch, species corresponding to all other branches become infinitely massive and decouple from the theory thereby reproducing the correct chiral anomaly. The branches other than the first one are rarely explored. However, recently the existence of an additional symmetry in the central branch which prohibits additive renormalization of fermion mass has been discovered in the ref. [7].

In this work, in order to explore all branches we introduce a generalized Wilson term containing a branch selector index (i_B). By choosing i_B one can make the fermions belonging to a particular branch physical. The fermions belonging to the rest of the branches become infinitely massive and decouple from the theory in the continuum limit. The conventional Wilson term corresponds to $i_B = 0$. To investigate the effect of radiative corrections, we calculate the additive mass

renormalization in fermion self-energy and the chiral anomaly to $\mathcal{O}(g^2)$ in perturbation theory for all the branches.

First we summarize the results of additive mass shift from tadpole and sunset contributions. The tadpole contributions for the first and fifth branches are equal in magnitude but opposite in sign. Same is true for the sunset contributions also. Thus δm vanishes if we average over the first and fifth branches. Similar observations hold for the second and the fourth branches also. Coming to the central branch the additive mass shifts from tadpole and sunset contributions separately vanish. This leads to the absence of additive mass renormalization in accordance with theoretical expectation. In the calculation of chiral anomaly first we perform an analytical calculation setting $am = 0$ and using the Karsten-Smit identity. We find the correct value of the anomaly for different branches with corresponding degeneracy factors and signs dictated by the chiral charges in the continuum limit. Since numerical simulations are performed at finite volume, finite lattice spacing and finite fermion mass, we have studied the effect of symmetry violation (given in eq. 2.4) on the anomaly integral as a function of the lattice fermion mass. The cut-off effects are almost equal in magnitude but opposite in sign for the first and the fifth branches. Same holds for the second and the fourth branches also. The cut-off effect is minimal for the central branch.

In conclusion, our exploration of the different branches of the fermion doublers in perturbation theory, in the context of additive mass renormalization and chiral anomaly, has shown that by appropriately averaging over suitably chosen branches one can reduce cut-off artifacts. Comparing the central branch with all other branches, we find that the central branch, among all the avatars of the Wilson fermion, is the most suitable candidate for exploring near conformal lattice field theories.

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